

DIFFERENCE DISCRETE CONNECTION AND CURVATURE ON CUBIC LATTICE

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Abstract

In a way similar to the continuous case formally, we define in different but equivalent manners the difference discrete connection and curvature on discrete vector bundle over the regular lattice as base space. We deal with the difference operators as the discrete counterparts of the derivatives based upon the differential calculus on the lattice. One of the definitions can be extended to the case over the random lattice. We also discuss the relation between our approach and the lattice gauge theory and apply to the discrete integrable systems.

Keyword: discrete connection, discrete curvature, noncommutative calculus, lattice gauge theory, discrete Lax pair

1 Introduction

The discrete systems play very important roles in various fields, so they are widely studied in different branches. As one of the successful ways to deal with the quantum gauge field theory non-perturbatively, the lattice gauge theory in high energy physics has opened up a new direction in both physics and mathematics in order to deal with the gauge potentials as the connections in a discrete manner. For the integrable systems, there are a kind of discrete integrable ones as the discrete counterparts of the continuous systems by means of the integrable discretization method. For the structure preserving algorithms or the geometric algorithms in computational mathematics, the continuous systems are discretized in such a way that some of the important properties such as the symplectic or multisymplectic structure, or the symmetry of the systems are required to be preserved in a discrete manner, respectively. There are many other important discrete systems and some of them are even without the proper or unique continuous limits. In this paper, we focus on how to get the discrete counterparts in a systematic manner for such a kind of continuous systems that there are important properties like the gauge potentials as connections, the symplectic or multisymplectic structures, the Lax pairs, the symmetries and so on.

A simple and direct method to get a discrete counterpart for a given continuous system such as an ODE or a PDE is to discrete the independent variable(s) and let the dependent variables become discrete accordingly without specially chosen. However, in most cases all important properties of the continuous system may be lost and the behaviors of the discrete system are even hard to be controlled. For constructing mostly quarried ones in all possible discretizations of the corresponding continuous systems, there is a working guide line or a structure-preserving criterion. Namely, it is important to look forward to those discrete systems that preserve as much as possible the intrinsically important properties of the continuous system (see, e.g. [1], [2], [3], [4], [5], [6]).

In the course of discretization, only a few of the most important properties, or “structure” could be maintained. Thus, it is needed to select these “structures”, to find

their discrete counterparts and to know how to preserve them discretely with the lowest price has to pay. For example, there are two classes of conservation laws in canonical conservative mechanics. The first class is of phase-area conservation laws characterized by the symplectic preserving property. Another class is related to energy and all first integrals of the canonical equations. Thus, it is needed to know if it is possible to establish such a kind of discrete systems that they not only discretely preserve the “structure”, such as the symplectic structure, but also the energy conservation discretely. And is it possible to get these discrete systems by a discrete variational principle?

In fact, as far as the discrete variation for the discrete mechanics are concerned, there are different approaches. In usual approach (see, e.g. [7], [8], [9], [10], [11]), only the discrete dependent variables are taken as the independent variational variables, while their differences are not. However, in the discrete variational principle proposed by two of the present authors and their collaborators recently [3], [4], [5], [6], the differences of the dependent variables as the discrete counterparts of derivatives are taken as the independent variational variables together with the discrete dependent variables themselves. Actually, this is just the discrete analogue of the variational principle for the continuous mechanics, where the derivatives of the dependent variables are dealt with as the independent variational variables first. Thus, the difference discrete Legendre transformation can be made and the method can be applied to either discrete Lagrangian mechanics or its Hamiltonian counterpart via the difference discrete Legendre transformation. The approach has been applied to the symplectic and multisymplectic algorithms in the both Lagrangian and Hamiltonian formalism. It has been also generalized to the case of variable steps in order to preserve the discrete energy in addition to the symplectic or the multisymplectic structure[6], [7], [8], [9], [12].

For the lattice gauge theory as the discrete counterpart of the gauge theory in continuous spacetime, the discrete gauge potentials, field strength and the action had been introduced in a manner almost completely different from, as least apparently, the ones in ordinary gauge fields or in the connection theory [13], [14] in fibre bundle. Although the discrete connection theory also has its own right (see, e.g. [15, 16]) as an application of the non-commutative geometry [17], how to introduce the discrete gauge potential in

the lattice gauge theory as a kind of discrete connection is an important and interesting problem. Two of us with their collaborator had considered this issue in [18] very briefly in a way different from other relevant proposals (see, e.g. [19], [20], [21], [22],[23]).

In this paper, we study the discrete connection and curvature on the regular lattice further in a way similar to the connection and curvature on vector bundle. We define them in different but equivalent manners, find their gauge transformation properties, the Chern class in the Abelian case and relevant issues. We show that the discrete connection and curvature introduced here are completely equivalent to the ones in the lattice gauge theory on the regular lattice. We also apply these issues to the discrete integrable systems and show that their discrete Lax pairs and the discrete-curvature free conditions are certainly similar to their continuous cases.

One of the key points in our approach is still to regard the difference operators acting on the functions space over the regular lattice as a kind of independent geometric objects and their dual should be the one forms such that we can introduce the discrete tangent bundle over the lattice and the cotangent bundle as its dual with the basis as the difference operators and one-forms, respectively. These are just the discrete counterparts of the continuous cases, where the derivatives and the one-forms as their dual play the roles as the basis of the tangent bundle and the cotangent bundle, respectively. In order to do so, the non-commutative differential calculus on the function space over the lattice [18] has to be employed. Similarly, the discrete vector bundle over the regular lattice can also be set up. In the connection theory à la Cartan, the exterior differential of the basis of a vector space at a point should be expanded in terms of the basis and the expanding coefficients are just the coefficients of the connection on the vector space. Since all counterparts of the basis, exterior differential and so on are equipped in the discrete vector bundle over the lattice, the discrete connection can also be introduced in a way similar to the one à la Cartan. This is our simple way to introduce the discrete connection. In continuous cases, there are several equivalence definitions for the connection and curvature. Similarly, we also introduce some of the equivalent definitions for the discrete connection and curvature on the lattice. It should be noted that the definition of the discrete connection in terms of the decomposition of the tangent space

of the discrete vector bundle may be written in the form without differences. Thus, it can be generalized to the cases over the random lattices.

The paper is organized as follows. In order to show the background and necessary preparation, we briefly recall the discrete mechanics and the non-commutative differential calculus on hypercubic lattice of high dimension in section 2 and section 3, respectively. In section 4, we discuss the discrete connection, curvature, Chern class and their gauge transformation properties. In section 5, we generalize one of the definitions for the discrete connection to the case of random lattice. Some applications to the lattice gauge theory and discrete integrable systems are given in section 6. We end with some remarks and discussions.

2 Difference Discrete Mechanics

In order to introduce the structure of discrete bundle, it is useful to review the formulism in the Lagrangian mechanics and the discrete mechanics as its discrete counterpart.

Let $t \in T$ be the time, M an n -dimensional configuration space as a vector space for simplicity. A particle moving on the configuration space is denoted in terms of its generalized coordinates as $q^i(t) \in M$ and its generalized velocities $\dot{q}^i(t) = dq^i(t)/dt$ as an element in tangent bundle TM of M . The space of all the possible path of particle moving in configuration space is an infinite dimensional space. The Lagrangian of a system is a functional defined on this space and denoted as $L(q^i(t), \dot{q}^i(t))$, $i = 1, \dots, n$. For simplicity, the Lagrangian in our discussion is of the first order and independent of t . The action functional is

$$S([q^i(t)]; t_1, t_2) = \int_{t_1}^{t_2} dt L(q^i(t), \dot{q}^i(t)). \quad (1)$$

Here $q^i(t)$ describes a curve C_a^b with ending points a and b , $t_a = t_1, t_b = t_2$, along which the motion of the system may takes place.

For the difference discrete Lagrangian mechanics, let us consider the case that “time”

t is difference discretized

$$t \in T \rightarrow t_k \in T_D = \{(t_k, t_{k+1} = t_k + \Delta t_k = t_k + h, \quad k \in Z)\} \quad (2)$$

and the step-lengths $\Delta t_k = h$ are equal to each other for simplicity, while the n -dimensional configuration space M_k at each moment $t_k, k \in Z$, is still continuous and smooth enough.

Let N be the set of all nodes on T_D with index set $Ind(N) = Z$, $M = \bigcup_{k \in Z} M_k$ the configuration space on T_D that is at least piecewise smooth enough. At the moment t_k , \mathcal{N}_k be the set of nodes neighboring to t_k . Let I_k the index set of nodes of \mathcal{N}_k including t_k . The coordinates of M_k are denoted by $q^i(t_k) = q^{i(k)}, i = 1, \dots, n$. $T(M_k)$ the tangent bundle of M_k in the sense that difference at t_k is its base, $T^*(M_k)$ its dual. Let $\mathcal{M}_k = \bigcup_{l \in I_k} M_l$ be the union of configuration spaces M_l at $t_l, l \in I_k$ on \mathcal{N}_k , $T\mathcal{M}_k = \bigcup_{l \in I_k} TM_l$ the union of tangent bundles on M_k , $F(TM_k)$ and $F(T\mathcal{M}_k)$ the functional spaces on each of them respectively, etc.. Sometime, it is also needed to include the links, plaquette and so on as well as the dual lattice, like in the lattice gauge theory, the mid-point scheme in the symplectic algorithm and so on. In these cases, the notations and conceptions introduced here should be generalized accordingly.

The above consideration should also make sense for the vector bundle over either the 1-dimensional lattice T_D or the higher dimensional lattice as a discrete base manifold. In the difference variational approach and the definition of the difference discrete connection, these notions may be used.

Now the discrete Lagrangian $L_D^{(k)}$ on $F(T(M_k))$ reads

$$L_D^{(k)} = L_D(q^{i(k)}, \Delta_k q^{i(k)}), \quad (3)$$

with the difference $\Delta_k q^{i(k)}$ of $q^{i(k)}$ at t_k defined by

$$\Delta_k q^{i(k)} := \frac{q^{i(k+1)} - q^{i(k)}}{t_{k+1} - t_k} = \frac{1}{h}(q^{i(k+1)} - q^{i(k)}). \quad (4)$$

The discrete action of the system is given by

$$S_D = \sum_{k \in Z} h \cdot L_D^{(k)}(q^{i(k)}, \Delta_k q^{i(k)}). \quad (5)$$

The discrete variation for $q^{i(k)} = q^i(t_k)$ should be defined as

$$\delta q^{i(k)} := q^i(t_k) - q^i(t_k). \quad (6)$$

And the discrete variations for $\Delta_k q^{i(k)}$ are given by

$$\delta \Delta_k q^{i(k)} = \Delta_k \delta q^{i(k)}. \quad (7)$$

Thus, the variations of the discrete Lagrangian can be calculated

$$\delta L_D^{(k)} = [L_{q^{i(k)}}] \delta q^{i(k)} + \Delta_k (p_i^{(k+1)} \delta q^{i(k)}), \quad (8)$$

where $[L_{q^{i(k)}}]$ is the discrete Euler-Lagrange operator

$$[L_{q^{i(k)}}] := \frac{\partial L_D^{(k)}}{\partial q^{i(k)}} - \Delta \left(\frac{\partial L_D^{(k-1)}}{\partial \Delta q^{i(k-1)}} \right), \quad (9)$$

and $p_i^{(k)}$ the discrete canonical conjugate momenta

$$p_i^{(k)} := \frac{\partial L_D^{(k-1)}}{\partial \Delta q^{i(k-1)}}. \quad (10)$$

And the variation of the discrete action can be written as

$$\delta S_D = \sum_k h [L_{q^{i(k)}}] \delta_v q^{i(k)} + \Delta (p_i^{(k+1)} \delta_v q^{i(k)}). \quad (11)$$

The variational principle requires $\delta S_D = 0$, so it follows the discrete Euler-Lagrange equations for $q^{i(k)}$'s

$$\frac{\partial L_D^{(k)}}{\partial q^{i(k)}} - \Delta \left(\frac{\partial L_D^{(k-1)}}{\partial \Delta q^{i(k-1)}} \right) = 0. \quad (12)$$

In order to transfer to the discrete Hamiltonian formalism, it is needed to introduce the discrete canonical conjugate momenta according to the equation (10) and express the discrete Lagrangian by the discrete Hamiltonian via Legendre transformation

$$H_D^{(k)} := p_i^{(k+1)} \Delta_t q^{i(k)} - L_D^{(k)}. \quad (13)$$

Thus, the discrete action can be expressed as

$$S_D = \sum_k h \cdot (p_i^{(k+1)} \Delta_t q^{i(k)} - H_D^{(k)}). \quad (14)$$

Now, Hamilton's principle requires $\delta_v S_D = 0$, it follows the discrete canonical equations for $p_i^{(k)}$'s and $q^{i(k)}$'s

$$\Delta q^{i(k)} = \frac{\partial H_D^{(k)}}{\partial p_i^{(k+1)}}, \quad \Delta p_i^{(k)} = -\frac{\partial H_D^{(k)}}{\partial q^{i(k)}}. \quad (15)$$

As was mentioned, the advantages of the difference discrete variational principle are based on keeping the difference operator as a discrete derivative operator. It is also clear that this approach can be applied to the field theory as well and generalized to the total discrete variation with variable step-lengths [6, 12]. Actually, this key point will also play a central role in our proposal to the discrete connection and curvature.

In the usual discrete variation, however, the $Q \times Q$ is used to indicate the vector field on the discrete space and the difference has not been dealt with as independent variables (see, e.g. [7], [8], [9], [24], [25], [26]). The corresponding discrete action is

$$S = \sum_{k=0}^{n-1} h \cdot \mathbb{L}(q_k, q_{k+1}), \quad (16)$$

where the Lagrangian $\mathbb{L}(q_k, q_{k+1})$ is the functional on $Q \times Q$. This is also the central idea of the resent proposal to the disconnection in [16]. Namely, using the tensor product $Q \times Q$ to study discrete tangent space of Q . In other words, the tangent vector $\dot{q}(t)$ at t_k is represented by a pair of nodes (q_k, q_{k+1}) without introducing the difference operator. Thus, the difference discrete Legendre transformation and discrete Hamiltonian formalism via the transformation cannot be formulated. In this case it is expected that the groupoid formulism may be used so that there is possibility to understand some geometric meanings of discrete models [27].

3 Difference and Differential Form on Lattice

In this section, we recall how to apply the differential calculus on discrete group [19] to the hypercubic lattice [18]. Although the result is similar to the one in ([20], [21], [22],[23]) the key point is different. In our approach, the shift operator is regarded as the generator of a discrete Abelian group in each direction of the high dimensional hypercubic lattice. For simplicity, we focus on the lattice with equal spacing $h = 1$.

Thus, the dimension of vector fields or differential forms is equal to the number of the shift operators of the lattice.

Let N and \mathcal{A} be a lattice and the algebra of complex valued functions on N , respectively, define the right and left shift operators E_μ, E_μ^{-1} at a node $x \in N$ in the μ -direction by

$$E_\mu x = x + \hat{\mu}, \quad E_\mu^{-1} x = x - \hat{\mu}, \quad (17)$$

and introduce a homeomorphism on the function space \mathcal{A} ,

$$E_\mu f(x) = f(x + \hat{\mu}), \quad E_\mu(f(x) \cdot g(x)) = E_\mu f(x) \cdot E_\mu g(x), \quad f, g \in \mathcal{A}, \quad (18)$$

where $(x - \hat{\mu})$, x and $(x + \hat{\mu})$ are the points on \mathcal{N}_x and they are the nearest neighbors on the μ -direction, the dot denotes the multiplication in \mathcal{A} .

The tangent space at the node x of \mathcal{N}_x is defined as $T\mathcal{N}_x := \text{span}\{\Delta_\mu|_x, \mu = 1, \dots, n\}$, where the operator Δ_μ is defined on the link between x and $x + \hat{\mu}$ and its action on \mathcal{A} is the differences in μ -th direction as,

$$\Delta_\mu f(x) := (E_\mu - id)f = f(x + \hat{\mu}) - f(x). \quad (19)$$

The above difference operator is a discrete analogue of a bases $\partial_\nu := \frac{\partial}{\partial x^\nu}$ for a vector field $X = X^\nu \partial_\nu$ in the continuous case.

The action of a difference operator Δ_μ in (19) on the functional space \mathcal{A} satisfies the deformed Leibnitz rule

$$\Delta_\mu(f(x) \cdot g(x)) = \Delta_\mu f(x) \cdot g(x + \hat{\mu}) + f(x) \Delta_\mu g(x). \quad (20)$$

For a given node $x \in \mathcal{N}_x$, all Δ_μ form a set of bases of the tangent space $T\mathcal{N}_x$. Its dual space denoted as $T^*\mathcal{N}_x$ is a space of 1-forms with a set of bases dx^μ defined on the link, too. They satisfy

$$dx^\mu(\Delta_\nu) = \delta_\nu^\mu,$$

which is also denoted as Ω^1 and $\Omega^0 = \mathcal{A}$ like the continuous case.

Thus, the tangent bundle and its dual cotangent bundle over N can be defined as

$$TN := \bigcup_{x \in \mathcal{N}} T\mathcal{N}_x \quad \text{and} \quad T^*N := \bigcup_{x \in N} T^*\mathcal{N}_x, \quad (21)$$

respectively.

Let us construct the whole differential algebra $\Omega^* = \bigoplus_{n \in \mathbb{Z}} \Omega^n$ on T^*N as in continuous case. The exterior derivative operator $d_D : \Omega^k \rightarrow \Omega^{k+1}$ is defined as

$$d_D \omega = \sum_{\alpha} \Delta_{\alpha} f dx^{\alpha} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \in \Omega^{k+1}, \quad (22)$$

where

$$\omega = f dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \in \Omega^k. \quad (23)$$

When $k = 0$, $\Omega^0 = \mathcal{A}$, then $d : \Omega^0 \rightarrow \Omega^1$ is given by

$$d_D f = \sum_{\alpha} \Delta_{\alpha} f dx^{\alpha}. \quad (24)$$

It is straightforward to prove that

$$\begin{aligned} (a) : \quad & (d_D f)(v) = v(f), \quad v \in T(N), f \in \Omega^0, \\ (b) : \quad & d_D(\omega \otimes \omega') = d_D \omega \otimes \omega' + (-1)^{\deg \omega} \omega \otimes d_D \omega', \quad \omega, \omega' \in \Omega^*, \\ (c) : \quad & d_D^2 = 0, \end{aligned} \quad (25)$$

provided that the following conditions hold

$$\begin{aligned} (1) \quad & dx^{\mu} \wedge dx^{\nu} = -dx^{\nu} \wedge dx^{\mu}, \\ (2) \quad & d_D(dx^{\mu}) = 0, \\ (3) \quad & dx^{\mu} f = (E_{\mu} f) dx^{\mu}, \quad (\text{no summation}). \end{aligned} \quad (26)$$

Thus, we set up the well defined differential algebra. Note that in order to do so the multiplication of functions and one-forms must be noncommutative.

4 Difference Discrete Connection and Curvature

The discrete analogue of connection has been given by two of the present authors [18] and others (see, e.g. [20], [21], [22], [23], [15], [16]). In this section we define the (difference) discrete connection in simple way similar to that in the continuous case based on the noncommutative differential calculus introduced in the last section. As was mentioned, the key point is to replace the difference discrete exterior derivative by the covariant difference discrete derivative for the sections on bundle.

4.1 Connection, Gauge Transformations and Holonomy

4.1.1 Discrete Bundle and Discrete Sections

Let $P = P(\mathcal{M}, G)$ be a principal bundle over an n -dimensional base manifold \mathcal{M} isomorphic to R^n with a Lie group G as the structure group. Now consider its discrete counterpart in the following manner as a discrete principle bundle. Let the manifold \mathcal{M} be discretized as \mathbb{Z}^n , i.e. the hypercubic lattice with equal spacing $h = 1$, and take such an $\mathcal{M} \simeq \mathbb{Z}^n$ (a lattice N) as the discrete base space. For a node $x \in N$, there is a fiber $G_x = \pi^{-1}(x)$ isomorphic to the Lie group G as the structure group. The union of all these fibers are called discrete principal bundle and denoted as $Q(N, G)$:

$$Q = \bigcup_{x \in N} \pi^{-1}(x). \quad (27)$$

If the structure group G is a linear matrix group, for example, $GL(m, R)$ or its subgroup, we may also define an associated vector bundle $V = V(N, R^m, GL(m, R))$ to the discrete principal bundle $Q(N, GL(m, R))$. For any $x \in N$ there is a fiber $F_x = \pi^{-1}(x)$ isomorphic to an m -dimensional linear space R^m with the right action of $GL(m, R)$ on the F_x . The union of all these fibers is called a discrete vector bundle:

$$V = \bigcup_{x \in N} \pi^{-1}(x). \quad (28)$$

The discrete field on discrete bundle is a section $h(x) \in G$ on discrete principal bundle or a $\psi^j(x) \in R^m$ on the vector bundle for all $x \in \mathcal{N}_x$. A section on discrete bundle is the map,

$$G : N = \bigcup \mathcal{N}_x \longrightarrow Q = Q(N, G), \quad G : x \mapsto G(x). \quad (29)$$

Similarly, we can define the discrete section on a discrete vector bundle. The union of the all sections is denoted as $\Gamma(Q)$ for the discrete principal bundle or as $\Gamma(V)$ for the discrete vector bundle, respectively.

One of the examples for the discrete vector bundle is the tangent bundle TN over an n -dimensional hypercubic lattice N in eq.(21). The base manifold of this bundle is the hypercubic lattice, the fiber $\pi^{-1}(x)$ over $x \in N$ is an n -dimensional vector space with

basis $\{\Delta_\mu, \mu = 1, \dots, n\}$. Another example is its dual bundle T^*N , its fiber is also the n -dimensional vector space with basis $\{dx^\mu, \mu = 1, \dots, n\}$.

For the discrete vector bundle the section space $\Gamma(V)$ is also the vector space. The same for the $\Gamma(T^*N)$. Their tensor product is given by

$$\Gamma(T^*N \times V) = \Gamma(T^*N) \times \Gamma(V). \quad (30)$$

As was mentioned in previous section, the one form basis dx^μ in discrete case is defined on a point $x \in N$ but links the point with another nearby point $x + \hat{\mu}$. Then the section of one forms in $\Gamma(T^*N)$ is generally defined on \mathcal{N}_x , its structure is very different from the section in continuous case. So we call it as discrete section. The section in $\Gamma(V)$ may be the discrete section as in $\Gamma(TN)$ of tangent vector bundle TN . However there is another possibility. Namely, the section is defined on the node only as in the discrete principal G -bundle. The direct product here can be simply understood as the discrete counterpart of the direct product in the continuous case in the above manner.

4.1.2 Definition of Connection and Gauge Transformations

Definition : A difference discrete connection or covariant difference discrete derivative is the linear map

$$\mathcal{D} : \Gamma(V) \longrightarrow \Gamma(T^*N \times V), \quad (31)$$

which satisfies the following condition:

$$\begin{aligned} \mathcal{D}(s_1 + s_2) &= \mathcal{D}s_1 + \mathcal{D}s_2, \\ \mathcal{D}(as) &= d_D a \otimes s + a\mathcal{D}s, \end{aligned} \quad (32)$$

where $s, s_1, s_2 \in \Gamma(V)$ and $a \in \mathcal{A}$. This is almost the same as the definition of the connection or the covariant derivative in the continuous case, which is basically equivalent to the connection a la Cartan.

Since our discussion on all geometric quantities are in the local sense, for the simplicity, we can choose the basis $\{s_\alpha, 1 \leq \alpha \leq m\}$ as the basis of the linear space $\Gamma(V)$ and $\{dx^\mu \otimes s_\alpha, 1 \leq \alpha \leq m, 1 \leq \mu \leq n\}$ as the basis of the sections space $\Gamma(T^*N \times V)$.

Then the covariant derivative $\mathcal{D}s_\alpha$ should be the linear expansion on $\{dx^\mu \otimes s_\alpha\}$. Hence, we can define it in the local sense as

$$\mathcal{D}s_\beta = \sum_{\alpha} \left[- (B)_{\beta}^{\alpha} \otimes s_{\alpha} \right] = \sum_{\alpha, \mu} \left[- (B_{\mu})_{\beta}^{\alpha} dx^{\mu} \otimes s_{\alpha} \right] \quad (33)$$

or simply

$$\mathcal{D}s = -B \cdot s, \quad (34)$$

where $B = B_{\mu} dx^{\mu}$ is the local expression of discrete connection 1-form. For the continuous case, the connection 1-form is valued on a Lie algebra. However the 1-form $B = B_{\mu} dx^{\mu}$ here is matrix valued. Since all 1-forms are defined on the links, the coefficients B_{μ} are also defined on the link $(x, x + \hat{\mu})$ and can be written as

$$B_{\mu}(x) = B(x, x + \hat{\mu}). \quad (35)$$

For any section $S = \sum_{\alpha} a^{\alpha} s_{\alpha}$, we have

$$\begin{aligned} \mathcal{D}S &= \sum (d_D a^{\alpha} \cdot s_{\alpha} + a^{\alpha} \cdot \mathcal{D}s_{\alpha}) \\ &= \sum (\Delta_{\mu} a^{\alpha} dx^{\mu} \otimes s_{\alpha} - a^{\alpha} (B_{\mu})_{\alpha}^{\beta} dx^{\mu} \otimes s_{\beta}) \\ &= \sum (\Delta_{\mu} a^{\beta} - a^{\alpha} (B_{\mu})_{\alpha}^{\beta}) dx^{\mu} \otimes s_{\beta} \\ &= \sum (\mathcal{D}_{D\mu} a^{\beta}) dx^{\mu} \otimes s_{\beta}, \end{aligned} \quad (36)$$

where

$$\mathcal{D}_{D\mu} a^{\beta} = \Delta_{\mu} a^{\beta} - a^{\alpha} (B_{\mu})_{\alpha}^{\beta} \quad (37)$$

is called the discrete covariant derivative of the vector a^{α} and

$$\mathcal{D}_D a^{\alpha} = \sum \mathcal{D}_{D\mu} a^{\alpha} dx^{\mu} = \sum (\Delta_{\mu} a^{\alpha} - a^{\beta} (B_{\mu})_{\beta}^{\alpha}) dx^{\mu} \quad (38)$$

is the discrete exterior covariant derivative of the vector a^{α} .

On the space $\Gamma(V)$, we can choose another linear basis or perform the linear transformation of the basis, i.e., take the gauge transformation as follows

$$s_{\alpha} \longmapsto \tilde{s}_{\alpha} = g(x)_{\alpha}^{\beta} \cdot s_{\beta}, \quad x \in N, \quad (39)$$

where $g(x)_\alpha^\beta$ is the $GL(m, R)$ gauge group valued function defined on the node $x \in N$. In order to keep the section S being invariant, the coefficients a^α of a general section $S = \sum a^\alpha s_\alpha$ should transform as

$$a^\alpha \longmapsto \tilde{a}^\alpha = a^\beta \cdot (g^{-1}(x))_\beta^\alpha. \quad (40)$$

From the gauge invariance of S and $\mathcal{D}S$, we can derive the gauge transformation property of $\mathcal{D}_{D\mu}a^\alpha$,

$$\begin{aligned} \mathcal{D}S &\longmapsto \sum (\mathcal{D}_{D\mu}a^\gamma) dx^\mu \cdot (g^{-1}(x))_\gamma^\alpha \otimes g(x)_\alpha^\beta \cdot s_\beta \\ &= \sum (\mathcal{D}_{D\mu}a^\gamma) \cdot (g^{-1}(x + \hat{\mu}))_\gamma^\alpha dx^\mu \otimes g(x)_\alpha^\beta \cdot s_\beta, \end{aligned} \quad (41)$$

where the noncommutative commutation relation between function and 1-form basis is used. Then the covariance of the covariant derivative follows

$$\mathcal{D}_{D\mu}a^\alpha \longmapsto \mathcal{D}_{D\mu}a^\gamma \cdot (g^{-1}(x + \hat{\mu}))_\gamma^\alpha. \quad (42)$$

Thus, we get the gauge transformation property of the difference discrete connection 1-form as

$$B_\mu(x) dx^\mu \longmapsto g(x) \cdot B_\mu(x) dx^\mu \cdot g^{-1}(x) + g(x) \cdot d_D g^{-1}(x), \quad (43)$$

or

$$B_\mu(x) \longmapsto g(x) \cdot B_\mu(x) \cdot g^{-1}(x + \hat{\mu}) + g(x) \cdot \Delta_\mu g^{-1}(x). \quad (44)$$

Together with the gauge transformation property of the coefficients of a vector field in (40), the gauge covariance of the derivative $\mathcal{D}_{D\mu}a^\gamma$ is confirmed.

4.1.3 Discrete Connection via Horizontal Tangent Vector

For the vector bundle, there is another definition of connection. It is based on the decomposition of the total tangent vector of the bundle into the horizontal and vertical parts. Then the horizontal tangent vector invariant under the right operation of the structure group also defines a connection. In fact, the horizontal tangent vector is nothing but the covariant derivatives. This definition can also be given in an analogous manner for the discrete case here.

Let us consider the discrete vector bundle over a discrete base manifold N as the regular lattice with the fiber as a smooth enough m -dimensional vector space V_x at $x \in N$, like the V_k used in section 2. As we discussed before, the basis of tangent space TV_x is

$$X_\alpha = \frac{\partial}{\partial a^\alpha}, \quad (1 \leq \alpha \leq m), \quad (45)$$

where a^α , $1 \leq \alpha \leq m$, are the coordinates of the fiber V_x . The basis of tangent vector on discrete regular lattice is

$$\Delta_\mu, \quad (1 \leq \mu \leq n). \quad (46)$$

Then the basis of total tangent space of discrete vector bundle is the union of (45) and (46).

Similar to the continuous case, the vector space tangent to the fiber, i.e. the linear combination of basis in (45), is a vertical subspace of the total tangent space of the discrete vector bundle. Its complementary vectors of the vertical subspace in the total tangent of vector bundle are horizontal and constitute the horizontal subspace. The basis of horizontal subspace is as follows,

$$X_\mu = \Delta_\mu - (B_\mu)_\beta^\alpha a^\beta \frac{\partial}{\partial a^\alpha}, \quad (1 \leq \mu \leq n). \quad (47)$$

In comparison with the definition of difference discrete connection, it is easy to see that the horizontal vector is nothing but the covariant derivative in (37).

This means that from the decomposition of tangent vector on the total bundle space we get the coefficients $B_\mu(x)$ of the discrete connection. For a given difference discrete connection or its coefficients $B_\mu(x)$, we can also get a decomposition of the total tangent vector space of bundle into vertical and horizontal parts sufficiently and necessarily. This shows that the difference discrete connection on discrete vector bundle is equivalent to a decomposition of the total tangent vector space into vertical and horizontal subspaces as above.

Similarly, we can define the basis of dual space for the decomposition, i.e., the basis of the vertical and horizontal 1-form space, respectively, as

$$\omega^\alpha = da^\alpha + (B_\mu)_\beta^\alpha a^\beta dx^\mu, \quad (1 \leq \alpha \leq m), \quad (48)$$

and

$$\omega^\mu = dx^\mu, \quad (1 \leq \mu \leq n). \quad (49)$$

They satisfy the following dual relation:

$$\omega^\alpha(X_\beta) = \delta_\beta^\alpha, \quad \omega^\alpha(X_\mu) = 0, \quad \omega^\mu(X_\beta) = 0, \quad \omega^\mu(X_\nu) = \delta_\nu^\mu.$$

4.1.4 Covariant Derivative and Parallel Transport

From the above discussions, we can get the difference discrete connection on discrete vector bundle through the definition of the absolute derivative or the horizontal tangent vector. Both lead to the difference discrete covariant derivative for the vectors,

$$\mathcal{D}_{D\mu} a^\beta = \Delta_\mu a^\beta - a^\alpha (B_\mu)_\alpha^\beta. \quad (50)$$

In terms of the relation between Δ_μ and E_μ , we obtain another expression for covariant derivative as,

$$\mathcal{D}_{D\mu} a^\gamma(x) = E_\mu a^\gamma(x) - a^\beta(x) \cdot [(B_\mu(x))_\beta^\gamma + \delta_\beta^\gamma], \quad (51)$$

where δ_β^α is the unit matrix. Then we obtain another expression for the coefficient of discrete connection,

$$U_\mu(x) = [(B_\mu(x))_\beta^\gamma + \delta_\beta^\gamma], \quad (52)$$

which is an element of some group, for example the group $GL(m, R)$ in our discussions, and connects the points x and $x + \hat{\mu}$. We can also call $U_\mu(x)$ as the discrete $GL(m, R)$ -connection and express it as

$$U_\mu(x) = U(x, x + \hat{\mu}). \quad (53)$$

From the second expression of the coefficient of connection and the definition of covariant derivative for the vectors, it follows the parallel transport of the section of vector a^β if its covariant derivative is zero

$$\mathcal{D}_{D\mu} a^\beta = \Delta_\mu a^\beta - a^\alpha (B_\mu)_\alpha^\beta = 0 \quad (54)$$

or

$$E_\mu a^\gamma(x) - a^\beta(x) \cdot (U_\mu(x))_\beta^\gamma = 0. \quad (55)$$

Namely,

$$a^\gamma(x + \hat{\mu}) = a^\beta(x) \cdot (U(x, x + \hat{\mu}))_\beta^\gamma. \quad (56)$$

This means that the parallel transport of the section of vector $a^\beta(x)$ along the path $x \mapsto x + \hat{\mu}$ is expressed as

$$a^\beta(x) \longmapsto a^\beta(x + \hat{\mu}) = a^\gamma(x) \cdot (U(x, x + \hat{\mu}))_\gamma^\beta. \quad (57)$$

It is shown that there is a parallel transport on the discrete bundle along the curve on discrete base manifold for a given discrete connection on discrete bundle.

Due to the discrete connection coefficient $U(x, x + \hat{\mu})$ as a group element, there are the following group properties of $U(x, x + \hat{\mu})$ along the path decomposition of $(x, x + \hat{\mu} + \hat{\nu})$ into $(x, x + \hat{\mu})$ and $(x + \hat{\mu}, x + \hat{\mu} + \hat{\nu})$,

$$U(x, x + \hat{\mu}) \cdot U(x + \hat{\mu}, x + \hat{\mu} + \hat{\nu}) = U(x, x + \hat{\mu} + \hat{\nu}). \quad (58)$$

From the inverse of the parallel transport, it also follows that

$$(U(x, x + \hat{\mu}))^{-1} = U(x + \hat{\mu}, x). \quad (59)$$

The coefficient of the discrete connection $U(x, x + \hat{\mu})$ determines the parallel transport not only on a vector bundle but also on a $GL(m, R)$ principal bundle. Therefore, it is also called a discrete $GL(m, R)$ connection.

4.1.5 Parallel Transport on Discrete Principal Bundle

The matrix structure group of discrete vector bundle can be generalized to any Lie group G and the coefficients of a connection are the group G -valued with the right operations. Thus, we can get a difference discrete G -valued connection on a discrete principal G -bundle over the lattice.

In this case, the concept of parallel transport can be extended to the discrete principal G -bundle. For a section $h(x)$, the parallel transport with respect to a G -valued connection $U(x, x + \hat{\mu})$ reads

$$h(x) \longmapsto h(x + \hat{\mu}) = h(x) \cdot U(x, x + \hat{\mu}). \quad (60)$$

All elements here belong to the Lie group G and multiplication should be the group multiplication. The above equation can be expressed as

$$h(x_0) \longmapsto h(x_1) = h(x_0) \cdot U(x_0, x_1), \quad (61)$$

or

$$h(x_1) \longmapsto h(x_0) = h(x_1) \cdot U(x_1, x_0), \quad (62)$$

which implies that

$$U(x_1, x_0) = U(x_0, x_1)^{-1}. \quad (63)$$

Similarly, the covariant derivative for the section $h(x)$ can be given as

$$\mathcal{D}_{D\mu}h(x) = E_\mu h(x) - h(x) \cdot U_\mu(x), \quad (64)$$

and the covariant exterior derivative as

$$\mathcal{D}_D h(x) = \mathcal{D}_{D\mu} h(x) dx^\mu = \sum_{\mu} (E_\mu h(x) - h(x) \cdot U_\mu(x)) dx^\mu. \quad (65)$$

4.2 Difference Discrete Curvature, Bianchi Identity and Abelian Chern Class

4.2.1 Difference Discrete Curvature

Based on the definition of the difference discrete connection 1-forms on discrete vector bundle, we can define the curvatures 2-forms similar to the continuous case [13], [14]

$$F = d_D B + B \wedge B. \quad (66)$$

If we assume $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$, it follows that

$$F_{\mu\nu}(x) = \triangle_\mu B_\nu(x) - \triangle_\nu B_\mu(x) + B_\mu(x) \cdot B_\nu(x + \hat{\mu}) - B_\nu(x) \cdot B_\mu(x + \hat{\nu}). \quad (67)$$

Under the continuous limit, it is easy to see that this curvature should be the same as the usual formula of curvature.

Since the definition is in a similar formulation except for the non-commutative exterior differential calculus, it is also easy to check the covariance property of the curvature under the gauge transformation (43) or (44) as follows

$$F(x) \longmapsto \tilde{F}(x) = g(x) \cdot F(x) \cdot g^{-1}(x), \quad (68)$$

or the covariance behavior of its components

$$F_{\mu\nu}(x) \longmapsto \tilde{F}_{\mu\nu}(x) = g(x) \cdot F_{\mu\nu}(x) \cdot g^{-1}(x + \hat{\mu} + \hat{\nu}). \quad (69)$$

It is important to see that from the non-commutative property of differential calculus on lattice the shifting operator appears in the covariance of discrete curvature. This is a main difference between the continuous case and the discrete one. And it may lead to more difficulties in discussion of the gauge covariance and invariance property of tensors in the discrete case.

4.2.2 Curvature via Holonomy

In continuous case, the curvature may naturally appear in the homology consideration. As the (difference) discrete counterpart, the (difference) discrete curvature may also be described based on the holonomy consideration.

Let us consider the square of the exterior covariant derivatives

$$\begin{aligned} & (\mathcal{D}_D)^2 h(x) \\ &= \mathcal{D}_D \mathcal{D}_{D\mu} h(x) dx^\mu \\ &= \mathcal{D}_D \sum_\mu (E_\mu h(x) - h(x) \cdot U_\mu(x)) dx^\mu \\ &= \sum_{\mu\nu} E_\nu (E_\mu h(x) - h(x) \cdot U_\mu(x)) dx^\mu \wedge dx^\nu \\ &\quad - \sum_{\mu\nu} (E_\mu h(x) - h(x) \cdot U_\mu(x)) dx^\mu \wedge U_\nu(x) dx^\nu \\ &= h(x) \sum_{\mu\nu} U_\mu(x) dx^\mu \wedge U_\nu(x) dx^\nu \\ &= h(x) \sum_{\mu\nu} U_\mu(x) \cdot U_\nu(x + \hat{\mu}) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} h(x) \sum_{\mu\nu} [U_\mu(x) \cdot U_\nu(x + \hat{\mu}) - U_\nu(x) \cdot U_\mu(x + \hat{\nu})] dx^\mu \wedge dx^\nu. \end{aligned} \quad (70)$$

This leads to another expression for the curvature 2-form with its coefficients

$$\begin{aligned} F &= U^2 = \sum_{\mu\nu} U_\mu(x) dx^\mu \wedge U_\nu(x) dx^\nu, \\ F_{\mu\nu} &= \frac{1}{2} [U_\mu(x) \cdot U_\nu(x + \hat{\mu}) - U_\nu(x) \cdot U_\mu(x + \hat{\nu})]. \end{aligned} \quad (71)$$

The zero curvature condition $F = 0$ is

$$U_\mu(x) \cdot U_\nu(x + \hat{\mu}) = U_\nu(x) \cdot U_\mu(x + \hat{\nu}), \quad (72)$$

or

$$U_\mu(x) \cdot U_\nu(x + \hat{\mu}) \cdot U_\mu^{-1}(x + \hat{\nu}) \cdot U_\nu^{-1}(x) = 1, \quad (73)$$

or

$$U(x, x + \hat{\mu}) \cdot U(x + \hat{\mu}, x + \hat{\mu} + \hat{\nu}) \cdot U(x + \hat{\mu} + \hat{\nu}, x + \hat{\nu}) \cdot U(x + \hat{\nu}, x) = 1. \quad (74)$$

The expressions in last formula is nothing but the holonomy group in geometry or the plaquette variable in lattice gauge theory. We will discuss them in the section 6 and show also that zero curvature condition is just the integrable condition for a discrete integrable system.

4.2.3 Bianchi Identity and Abelian Chern Class

Similar to the Bianchi identity in differential geometry, we can also derive the Bianchi identity for the difference discrete curvature

$$\mathcal{D}_D F = d_D F - F \wedge B + B \wedge F = 0, \quad (75)$$

or in its components

$$\varepsilon^{\lambda\mu\nu} [\triangle_\lambda F_{\mu\nu}(x) - F_{\lambda\mu}(x) \wedge B_\nu(x + \hat{\mu} + \hat{\nu}) + B_\lambda(x) \wedge F_{\mu\nu}(x + \hat{\lambda})] = 0. \quad (76)$$

For the Abelian case, one can define the following topological term as discrete Chern class [28], [29], [30] ,

$$c_k = F \wedge F \wedge \cdots \wedge F, \quad (77)$$

which was used to discuss the chiral anomaly in the lattice gauge theory. The coefficient of the Abelian Chern class is

$$\varepsilon^{\mu_1 \mu_2 \cdots \mu_{2k-1} \mu_{2k}} F_{\mu_1 \mu_2}(x) \cdot F_{\mu_3 \mu_4}(x + \hat{\mu}_1 + \hat{\mu}_2) \cdots F_{\mu_{2k-1} \mu_{2k}}(x + \hat{\mu}_1 + \hat{\mu}_2 + \cdots + \mu_{2k-2}).$$

This equation was first appeared in lattice gauge theory for the Abelian anomaly of chiral fermion in a quantum field theory [28],[29], [30].

5 Discrete Connection on G -Bundle over Random Lattice

The definition of the discrete connection via the horizontal vector space in sect. 4.1.3 can be generalized to the one on a G -bundle $Q(N, G)$ over a random lattice N .

Let us consider the parallel transport of a section on such a G -bundle:

$$h(x_0) \longmapsto h(x_1) = h(x_0) \cdot U(x_0, x_1), \quad (78)$$

where $h(x_j)$ is the G -valued section defined on x_j , $j = 0, 1$ and x_0, x_1 are nearest neighbor. We can reexpress equivalently it as

$$(x_0, h_0) \longmapsto (x_1, h_1) = (x_0, h_0) \cdot U(x_0, x_1), \quad (79)$$

where $h_0 = h(x_0)$, $h_1 = h(x_1)$ and right multiplication of U on the bundle acts only on the G -valued section h_0 . It is easy to see that in these expressions there is no difference operator involved so that they could be make sense for the G -bundle over random lattice, if the discrete connection is properly introduced.

On the other hand, if h_0 and h_1 satisfy eq.(78), it can be proved that the element $(q_0, q_1) = ((x_0, h_0), (x_1, h_1)) \in Q \times Q$ is just a horizontal vector on TQ , i.e.

$$\text{hor}((x_0, h_0), (x_1, h_1)) = ((x_0, h_0), (x_1, h_1)), \quad (80)$$

where $\text{hor}(*, *)$ denotes the horizontal part of the $(*, *)$. In fact, this is almost the same as the one introduced in [16].

Thus, our definition for the discrete connection can be easily compared with the local expression $A(x_0, x_1)$ of the coefficients of a connection 1-form defined in [16]. Namely,

$$U(x_0, x_1) = A(x_0, x_1)^{-1} \quad (81)$$

then

$$((x_0, h_0), (x_1, h_0 \cdot A(x_0, x_1)^{-1}))$$

is a horizontal vector. According to the formulation in [16], we get

$$\begin{aligned}
& ((x_0, h_0), (x_1, h_0 \cdot A(x_0, x_1)^{-1})) \\
&= h_0 \cdot i_{(x_0, e)}(A(x_0, x_1)^{-1}) \cdot ((x_0, e), (x_1, e)) \\
&= h_0 \cdot \text{hor}((x_0, e), (x_1, e)) \\
&= h_0 \cdot \text{hor}((x_0, e), q_1) \\
&= h_0 \cdot \text{hor}((x_0, e), h_0^{-1} q_1) \\
&= \text{hor}((x_0, h_0), q_1),
\end{aligned} \tag{82}$$

which means that the horizontal vector $((x_0, h_0), (x_1, h_0 \cdot A(x_0, x_1)^{-1}))$ is the horizontal part of any vector (q_0, q_1) with q_0 is fixed and q_1 is any point on the fiber of $\pi^{-1}(x_1)$.

According to the definition of $A(x_0, x_1)$, we have

$$A(x_0, x_1) = \mathcal{A}_d(x_0, e, x_1, e).$$

From the gauge transformation property of $U(x_0, x_1)$, it follows that under the gauge transformation $g(x)$

$$A(x_0, x_1) \longmapsto g(x_1) \cdot A(x_0, x_1) \cdot g^{-1}(x_0). \tag{83}$$

This leads to

$$\mathcal{A}_d(x_0, g(x_0); x_1, g(x_1)) = g(x_1) \mathcal{A}_d(x_0, e, x_1, e) g^{-1}(x_0). \tag{84}$$

Thus, we recover the property of the connection 1-form defined in [16]

$$\mathcal{A}_d(gq_0, hq_1) = h \mathcal{A}_d(q_0, q_1) g^{-1}. \tag{85}$$

As was shown above, our definition of discrete connection is equivalent to that in [16] in the case of the cubic lattice. However, our definition for the discrete curvature is only for the hypercubic lattice, since it is based on the noncommutative differential calculus. How to extend those results to the case of random lattice is under investigation.

6 Applications

6.1 Lattice Gauge Theory and Difference Discrete Connection

In the lattice gauge theory [31], the space-time is discretized as hypercubic lattice with equal spacing a in any direction in most cases.

Suppose that A_μ is the gauge field or the connection on the continuous case. At each link on the lattice we introduce a discrete version of the path ordered product

$$U(x, x + \hat{\mu}) \equiv U_\mu(x) = e^{iaA_\mu(x + \frac{\hat{\mu}}{2})}, \quad (86)$$

where $\hat{\mu}$ is the vector in coordinate direction with length a and x is the point coordinates on the node of the hypercubic lattice which takes integer value only. The average field, which is denoted by $A_\mu(x + \frac{\hat{\mu}}{2})$, is defined at the midpoint of the link $(x, x + \hat{\mu})$. Similarly,

$$U(x, x - \hat{\mu}) \equiv U_{-\mu}(x) = e^{-iaA_\mu(x - \frac{\hat{\mu}}{2})} = U^\dagger(x - \hat{\mu}, x). \quad (87)$$

If the connection A_μ is valued on the Lie algebra of $SU(N)$ with a hermitian basis, we have

$$U^\dagger(x - \hat{\mu}, x) = U^{-1}(x - \hat{\mu}, x).$$

The variable of a simplest Wilson loop called plaquette variable is expressed as the left side of (74), which is defined on the two dimensional square

$$W_{\mu\nu} = U_\mu(x) \cdot U_\nu(x + \hat{\mu}) \cdot U_\mu^\dagger(x + \hat{\mu} + \hat{\nu}) \cdot U_\nu^\dagger(x). \quad (88)$$

It can be shown that the continue limit of $W_{\mu\nu}$ is related to the Yang-Mills action

$$\text{Re}(1 - W_{\mu\nu}) = \frac{a^4}{2} F_{\mu\nu} F^{\mu\nu} + O(a^6) + \dots,$$

$$\text{Im}(W_{\mu\nu}) = a^2 F_{\mu\nu} + \dots.$$

Therefore, plaquette variable $W_{\mu\nu}$ should play some rule of curvature in the discrete case as we discussed in previous section. However its continuous limit is related not only to the usual curvature but also to the Yang-Mills action as in the above expressions, there should be more geometric meaning in the theory of discrete connection and curvature than usual one.

6.2 Geometric Meaning of Discrete Lax Pair

In order to understand the discrete connection on discrete bundle, we first discuss some geometric meanings of the Lax pair and discrete Lax pair. In fact, it gives one of solid motivations and some consideration for the study of discrete connection.

Let us start with the concept of Lax pair of integral system in continuous two dimensions case with 1-dimension time and 1-dimension space as follows

$$\begin{aligned}\partial_x \psi &= \psi \cdot A_x, \\ \partial_t \psi &= \psi \cdot A_t,\end{aligned}\tag{89}$$

where ψ is a vector and A_x, A_t are matrix valued. The consistent condition for this linear system is

$$\partial_x A_t - \partial_t A_x + [A_x, A_t] = 0.\tag{90}$$

Now let us discretize the 2-dimensional space-time as a square lattice, $R^2 \rightarrow \mathbb{Z}^2$. The section field $\psi(x)$ on the vector bundle becomes the field $\psi(m, n)$ the functions depending on two discrete variable, i.e., two integer (m, n) . Naively the discrete Lax pair may be written as

$$\begin{aligned}\Delta_x \psi(m, n) &= \psi(m, n) \cdot A_x(m, n), \\ \Delta_t \psi(m, n) &= \psi(m, n) \cdot A_t(m, n),\end{aligned}\tag{91}$$

The derivatives ∂_x and ∂_t with respect to x and t are replaced by difference operators Δ_x and Δ_t , respectively. The consistent condition for the discrete Lax pair, i.e.,

$$\Delta_x \Delta_t \psi(m, n) = \Delta_t \Delta_x \psi(m, n)\tag{92}$$

leads to

$$\Delta_x A_t(m, n) - \Delta_t A_x(m, n) + A_x(m, n)A_t(m+1, n) - A_t(m, n)A_x(m, n+1) = 0.\tag{93}$$

Using the shift operator E_x and E_t we can also rewrite the discrete Lax pair (91) as

$$\begin{aligned}E_x \psi(m, n) &= \psi(m, n) \cdot [1 + A_x(m, n)] = \psi(m, n) \cdot U_x(m, n) \\ E_t \psi(m, n) &= \psi(m, n) \cdot [1 + A_t(m, n)] = \psi(m, n) \cdot U_t(m, n),\end{aligned}\tag{94}$$

where

$$U_t(m, n) = 1 + A_t(m, n), \quad U_x(m, n) = 1 + A_x(m, n). \quad (95)$$

On requiring the corresponding consistent condition

$$E_x E_t \psi(m, n) = E_t E_x \psi(m, n), \quad (96)$$

a straightforward calculation leads to

$$U_x(m, n) \cdot U_t(m+1, n) = U_t(m, n) \cdot U_x(m, n+1), \quad (97)$$

or

$$U_x(m, n) \cdot U_t(m+1, n) \cdot U_x^{-1}(m, n+1) \cdot U_t^{-1}(m, n) = 1. \quad (98)$$

If we use the relation of U and A in (95), we can derive the zero curvature condition (93) form (98).

When we regard the quantities A_x , A_t , U_x and U_t as the discrete connections on discrete bundle, the equations (93) and (98) should be the zero curvature condition for these connections, and the left sides of (93) and (98) should be the extension of the curvature in the discrete case.

7 Remarks and Discussions

As was mentioned previously, the study of discrete models is very important in both their own right and applications, although we mainly focus on the discrete models as the discrete counterparts of the continuous cases.

In order to get the discrete models that can keep the properties of continuous ones as much as possible, we may first consider a kind of discrete models from their continuous counterparts with differences as discrete derivatives. These models can be given by replacing the both continuous independent variables and their derivatives by the discrete independent variables as a regular lattice and their differences on the lattice, respectively. In general, for the discrete models on the regular lattices including the models just mentioned, it is natural to study first the properties of the function spaces on the lattices and the discrete bundles over the lattices both analytically and geometrically,

such as discrete differential calculus, discrete metric, discrete Hodge operator, discrete connection and curvature, and so on. In doing so, we may follow a way similar to that in the continuous cases, as long as the differences are regarded as the discrete derivatives.

In this paper, we have briefly reviewed the non-commutative differential calculus on hypercubic lattice, which have discussed by many groups. We have mainly introduced the (difference) discrete connections on discrete vector bundle in several manners, the parallel transport, the decomposition of the vector space into vertical and horizontal space, the covariant derivative on the section of vector bundle as well as the discrete curvature of the discrete connections. We have also studied their relation to the lattice gauge theory and applied to the Lax pairs for the discrete integrable systems.

There are, of course, also many properties of the discrete models, which are very different apparently from the continuous ones. These should be investigated further. Although one of the definitions for the discrete connection can be extended to the case over the random lattice, for the discrete curvature on the random lattice in the lattice gauge theory, however, it is still open whether it can be defined in a way similar to the continuous case formally. This is also a very interesting question to get more results on lattice with non-trivial topology. Another very important problem is how to get the topological classes with non-Abelian group. Needless to say, more attention should be paid to those questions.

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